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Identification of the ten inertia parameters of a rigid body☆

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ABSTRACT

A special antisymmetric 4×4 matrix form of the equation of motion of a rigid body is proposed. This form depends linearly on the symmetric (4×4)-matrix of the Fayet global inertia tensor, containing the ten inertia parameters of a rigid body (the mass, the three coordinates of the centre of mass and the six components of the classical inertia tensor). For identifying the global inertia tensor, an algorithm is proposed which is based on the method of least squares and the method of conjugate gradients and tested using the example of a rigid body, the motion of which is obtained by computer modelling.

A knowledge of the inertia parameters of a rigid body, that is, the mass, the three coordinates of the centre of mass and the six components of the central inertia tensor is required in different applications and, in particular, for the precise control of motion and in inverse dynamic problems. Frequently, these parameters are only approximately known. In proportional models,[1–3] the inertia parameters of segments of the human body are determined by extrapolation from regression equations, determined for a certain sample. Inertia parameters can also be obtained by simulation of these segments by means of more or less complex geometrical figures.[4,5]

Identification procedures have been actively developed in robotics since, in the case of certain types of robots, there are no other routes for obtaining the required parameters.[6] The recursive Newton–Euler method has been used in which the inertia parameters are represented by a ten-dimensional vector[7] and it has been shown that, when identifying the inertia parameters of segments of the human body, on account of the high number of degrees of freedom, the results are found to be unsatisfactory (starting from six degrees of freedom).

In the identification algorithm below, the ten inertia parameters of a rigid body are united into a symmetric 4×4 matrix of the Fayet global inertia tensor[8,9] and the equations of motion are written using antisymmetric 4×4 matrices.[10]

1. Dynamic characteristics of a rigid body

Suppose an absolutely rigid body S moves in a Galilean reference system $Ox_0y_0z_0$ and that $A \in S$ is a certain fixed point in the body with an initial position A_0. We define the translational displacement of the body by the vector $T = \overline{OA}$ and rotation is described by the rotation matrix $R : \overline{AB} = R(t)\overline{A_0B_0}$ for an arbitrary point $B \in S$ with an initial position B_0.

Suppose $a(B)$ is the acceleration of point B. We shall call the time derivative of the momentum of the body the *dynamic resultant* χ and the derivative of the angular momentum with respect to point A the *dynamic moment* $\delta(A)$:

$$\chi = \int a(B)dm, \quad \delta(A) = \int [a(B) \otimes \overline{AB} - \overline{AB} \otimes a(B)]dm \tag{1.1}$$

The moments are henceforth represented by antisymmetric 3×3 matrices which are calculated using tensor multiplication and, unless otherwise stated, integration is carried out over $B \in S$.

Using the translation vector $T(t)$ and the rotational matrix $R(t)$, the position and acceleration of a point B at any instant of time can be expressed using the formulae

$$\overline{OB} = \overline{OA} + \overline{AB} = T(t) + R(t)\overline{A_0B_0}, \quad a(B) = \ddot{T}(t) + \ddot{R}(t)\overline{A_0B_0} \tag{1.2}$$

For simplicity, we shall henceforth omit the argument t in the functions $\ddot{T}(t)$, $R(t)$ and $\ddot{R}(t)$.

Let m be the mass of the body and G be its centre of mass. From relations (1.1) and (1.2) and the obvious equality $\chi = ma(G)$, we obtain

$$\chi = m\ddot{T} + \ddot{R}m\overline{A_0G_0}, \quad \delta(A) =$$
$$= \int [(\ddot{T} + \ddot{R}\overline{A_0B_0}) \otimes (R\overline{A_0B_0})]dm - \int [(R\overline{A_0B_0}) \otimes (\ddot{T} + \ddot{R}\overline{A_0B_0})]dm \tag{1.3}$$

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Using the property of transposition and the definition of the centre of mass

$$\ddot{T} \otimes (R\overline{A_0B_0}) = (\ddot{T} \otimes \overline{A_0B_0})R^T, \quad \int \overline{A_0B_0}dm = m\overline{A_0G_0}$$

we rewrite the second relation of (1.3) in the form

$$\delta(A) = (\ddot{T} \otimes m\ddot{R}\overline{A_0G_0})R^T - R(m\overline{A_0G_0} \otimes \ddot{T}) + \ddot{R}K_0R^T - RK_0\ddot{R}^T;$$
$$K_0 = \int(\overline{A_0B_0} \otimes \overline{A_0B_0})dm \tag{1.4}$$

Here K_0 is the Poinsot inertia tensor of the body S at the point A_0.

Remarks.

1°. The dynamic resultant χ depends linearly on the parameters m and $m\overline{A_0G_0}$, and the dynamic moment $\delta(A)$ depends linearly on the parameters $m\overline{A_0G_0}$ and K_0.
2°. The classical Poinsot inertia tensor I and the Poinsot tensor K are related by the equality $K = \mathrm{tr}IE/2 - I$, where E is a unit 3×3 matrix.

2. The equations of motion

Let F be the sum of the external forces acting on the body S, f is their mass density and $M(A)$ is the moment of the external forces about the point A:

$$F = \int f(B)dm, \quad M(A) = \int[f(B) \otimes \overline{AB} - \overline{AB} \otimes f(B)]dm$$

Separating out the force of gravity with the gravitational g from the system of forces, we put

$$F = \tilde{F} - mg, \quad M(A) = \tilde{M}(A) - g \otimes m\overline{AG} + m\overline{AG} \otimes g$$

On the basis of the general theorems of dynamics

$$\chi = F, \quad \delta(A) = M(A)$$

and, using relations (1.3) and (1.4), we rewrite the equations of motion in the vector form

$$m(\ddot{T} + g) + \ddot{R}m\overline{A_0G_0} = \tilde{F}$$
$$[(\ddot{T} + g) \otimes m\overline{A_0G_0}]R^T - R[m\overline{A_0G_0} \otimes (\ddot{T} + g)] + \ddot{R}K_0R^T - RK_0\ddot{R}^T = \tilde{M}(A) \tag{2.1}$$

Introducing the 4×4 matrices

$$\Omega = \begin{Vmatrix} R & 0 \\ 0^T & 1 \end{Vmatrix}, \quad \Theta = \begin{Vmatrix} \ddot{R} & \ddot{T} + g \\ 0^T & 0 \end{Vmatrix}, \quad \Gamma(A) = \begin{Vmatrix} \tilde{M}(A) & \tilde{F} \\ -\tilde{F}^T & 0 \end{Vmatrix},$$
$$H = \begin{Vmatrix} K_0 & m\overline{A_0G_0} \\ m\overline{A_0G_0}^T & m \end{Vmatrix}$$

we rewrite Eq. (2.1) in the matrix form

$$\Theta(t)H\Omega^T(t) - \Omega(t)H\Theta^T(t) = \Gamma(A, t) \tag{2.2}$$

with antisymmetric 4×4 matrices on the left and right-hand sides of the equation. The matrix H represents the Fayet global inertia matrix[8,9] and contains the ten inertia parameters of the body S. Equation (2.2) are convenient for the identifying the matrix H since they are linear with respect to H.

The matrices Ω and Θ contain information on the kinematic state of the body which is experimentally determined using cameras which record the motion of a body with reflecting markers fastened to it. The matrix $\Gamma(A)$ contains the forces and moments about the point A acting on a body. The information for this matrix is obtained using a force platform which measures the forces and moments acting on the body.

Remarks.

1°. The matrix form of the equations of motion contains six scalar equations corresponding to the six independent elements of an antisymmetric 4×4 matrix.
2°. The matrix Ω is a 4×4 rotation matrix in R^4 since $\Omega\Omega^T = \Omega^T\Omega = E$.

We will now show that the matrix H is positive definite. Actually, we rewrite it in the form

$$H = \int \begin{Vmatrix} \overline{A_0B_0} \\ 1 \end{Vmatrix} \otimes \begin{Vmatrix} \overline{A_0B_0} \\ 1 \end{Vmatrix} dm_0$$

from which the positiveness of the associated quadratic form

$$\langle V, HV \rangle = \int \left(\begin{Vmatrix} \overline{A_0B_0} \\ 1 \end{Vmatrix} \begin{Vmatrix} U \\ \upsilon \end{Vmatrix} \right)^2 dm_0 \geq 0; \quad V = \begin{Vmatrix} U \\ \upsilon \end{Vmatrix},$$
$$U \in R^3, \quad \upsilon \in R$$

follows for any vector V from R^4 ($\langle , \rangle$ is a scalar product).

It follows from the equality $\langle V, HV \rangle = 0$ that

$$\langle \overline{A_0B_0}, U \rangle + \upsilon = 0 \ \forall B_0 \in S$$

When $B_0 = A_0$, we obtain $v = 0$ and, for any $B_0 \in S$, it follows from the equality $\langle \overline{A_0B_0}, U \rangle = 0$ that $U = 0$ in the case of a three-dimensional body. Hence, $\langle V, HV \rangle = 0 \Rightarrow U = 0$ and the positive definiteness of the matrix H is proved.

Remark. The positive definiteness of the matrix H generalizes the positive definiteness of the Poinsot inertia tensor K_0.

3. The method of least squares

The following problem is considered: it is required to find the positive definite 4×4 matrix H which satisfies Eq. (2.2), knowing the 4×4 matrices $\Omega(t)$ and $\Theta(t)$, obtained by observing the motion of the body and, also, knowing the matrix $\Gamma(A, t)$, composed of the forces and the moments of the forces acting on the body.

We consider the space of the matrices $M_{4\times4}$ with a scalar product $\langle A, B \rangle = \mathrm{tr}(AB^T)$ for any $A, B \in M_{4\times4}$. In the subspace $\tilde{M}_{4\times4}$ of symmetric 4×4 matrices, this scalar product reduces to the relation $\langle A, B \rangle = \mathrm{tr}(AB)$.

For each instant of time, Eq. (2.2) gives six conditions for determining the ten unknown parameters, and this equation must therefore be considered for at least two different instants of time. However, in order to average the noise, it is desirable that more than two instants of time are considered. We shall consider n experiments and suppose that t_i is the duration of the i-th experiment.

The method of least squares

$$\min\{J(H)|H \in \tilde{M}_{4\times 4}\}$$

is used to solve this overdetermined system of equations with a functional of the form

$$J(H) = \frac{1}{4}\sum_{i=1}^{n}\int_0^{t_i}\left\|\Theta_i(t)H\Omega_i^T(t) - \Omega_i(t)H\Theta_i^T(t) - \Gamma_i(A,t)\right\|^2 dt;$$

$$\forall Q \in M_{4\times 4},\ \|Q\|^2 = \langle Q, Q\rangle$$

For simplicity, we shall subsequently write Ω_i, Θi, $\Gamma_i\,(A)$ instead of $\Omega_i(t)$, $\Theta_i(t)$, $\Gamma_i(A, t)$. Using the property $\mathrm{tr}(PQ) = \mathrm{tr}(QP)$ for any P, $Q \in M_{4\times 4}$, we rewrite the functional in the form

$$J(H) = -\frac{1}{4}\sum_{i=1}^{n}\int_0^{t_i}\mathrm{tr}[(\Theta_i H\Omega_i^T - \Omega_i H\Theta_i^T - \Gamma_i(A))^2]dt =$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\int_0^{t_i}\mathrm{tr}(\Theta_i H\Omega_i^T\Theta_i H\Omega_i^T)_2 dt + \frac{1}{2}\sum_{i=1}^{n}\int_0^{t_i}\mathrm{tr}(H\Omega_i^T\Omega_i H\Theta_i^T\Theta_i)_2 dt +$$

$$+\sum_{i=1}^{n}\int_0^{t_i}\mathrm{tr}(H\Omega_i^T\Gamma_i(A)\Theta_i)_1 dt - \frac{1}{4}\sum_{i=1}^{n}\int_0^{t_i}\mathrm{tr}(\Gamma_i^2(A))_0 dt$$

The subscript after the brackets indicates the degree of homogeneity of the expression with respect to the components of the matrix H.

We will now show that the functional $J(H)$ is strictly convex and, consequently, its minimum is attained when the gradient vanishes:

$$-\nabla J(H) = \sum_{i=1}^{n}\int_0^{t_i}\frac{\Omega_i^T\Theta_i H\Omega_i^T\Theta_i + \Theta_i^T\Omega_i H\Theta_i^T\Omega_i}{2}dt -$$

$$-\sum_{i=1}^{n}\int_0^{t_i}\frac{\Theta_i^T\Theta_i H\Omega_i^T\Omega_i + \Omega_i^T\Omega_i H\Theta_i^T\Theta_i}{2}dt -$$

$$\sum_{i=1}^{n}\int_0^{t_i}\frac{\Omega_i^T\Gamma_i(A)\Theta_i + \Theta_i^T\Gamma_i(A)\Omega_i}{2}dt = 0$$

For the proof, we consider the linear mapping of symmetric 4×4 matrices in the space $\tilde{M}_{4\times 4}$

$$L(\cdot)\colon \tilde{M}_{4\times 4} \to \tilde{M}_{4\times 4}$$

$$H \mapsto \sum_{i=1}^{n}\int_0^{t_i}\left(-\frac{\Omega_i^T\Theta_i H\Omega_i^T\Theta_i + \Theta_i^T\Omega_i H\Theta_i^T\Omega_i}{2} + \frac{\Theta_i^T\Theta_i H\Omega_i^T\Omega_i + \Omega_i^T\Omega_i H\Theta_i^T\Theta_i}{2}\right)dt$$

The mapping $L(\cdot)$ is self-adjoint in $\tilde{M}_{4\times 4}$ in the sense of the scalar product $\langle\cdot,\cdot\rangle$ and positive, which follows from the relation

$$\langle L(H), H\rangle = \mathrm{tr}(L(H)H) = \sum_{i=1}^{n}\int_0^{t_i}\left\|\Theta_i H\Omega_i^T - \Omega_i H\Theta_i^T\right\|^2 dt > 0$$

The strict convexity of the functional follows from this.

4. The identification algorithm

The algorithm of the conjugate gradient method, which enables one to identify the matrix H, consists of the following successive steps.

Step 0. We select an initial approximation H_0 to the symmetric 4×4 matrix H and put $p = 0$ and $D_0 = -G_0 = -\nabla J(H_0)$. If $|G_0| < \varepsilon$, we terminate the calculations.

Step 1. We calculate $\mu_p > 0$ using the formula

$$\mu_p = \frac{\mathrm{tr}(G_p^T D_p)}{\mathrm{tr}\left[D_p\sum_{i=1}^{n}\int_0^{t_i}(\Omega_i^T\Theta_i D_p\Omega_i^T\Theta_i - \Omega_i^T\Omega_i D_p\Theta_i^T\Theta_i)dt\right]}$$

Step 2. We calculate

$$H_{p+1} = H_p + \mu_p D_p,\quad G_{p+1} = \nabla J(H_{p+1})$$

If $|G_{p+1}| < \varepsilon$, we terminate the calculation.

Step 3. We calculate

$$\gamma_p = \frac{\mathrm{tr}[G_{p+1}^T(G_{p+1} - G_p)]}{\mathrm{tr}(G_p^T G_p)},\quad D_{p+1} = -G_{p+1} + \gamma_p D_p$$

We increase the value of the index p by unity and return to step 1.

Remarks.

1°. The quantity μ, determined in Step 1, is obtained by solving the equation

$$\partial J(H + \mu D)/\partial\mu = 0$$

2°. The choice of γ_p in Step 3 gives the same result as

$$\gamma_p = \mathrm{tr}(G_{p+1}^T G_{p+1})/\mathrm{tr}(G_p^T G_p)$$

since the symmetric 4×4 matrix G_p is orthogonal by construction.

5. Example

We will now consider the case of the motion of a homogeneous parallelepiped of mass $m = 0.8$ kg and edges 0.4 m, 0.3 m and 0.5 m under the action of a force F applied at the point A (see Fig. 1).

From experiment to experiment, only the magnitude of the force F changes in the matrix

$$\Gamma(A) = \begin{Vmatrix} 0 & F \\ -F^T & 0 \end{Vmatrix}$$

The motion of the parallelepiped was simulated using the "Solid Dynamics" program. This program enables one to determine the kinematics of any point of a rigid body. In each experiment, the positions of the vertices B and C of the parallelepiped (Fig. 1) were

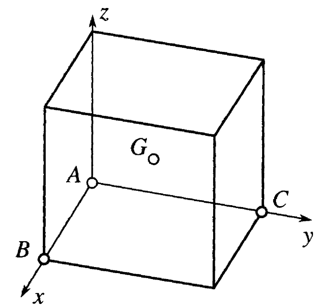

Fig. 1.

recorded at each instant of time. Using points A, B and C, the position of the trihedron $Axyz$ associated with the parallelepiped was determined and, in the final analysis, the matrices Ω and Θ were determined. The values H_1 and H_5 of the matrix H, obtained by identification using one and five simulations respectively, are presented below.

$$H_1 = \begin{Vmatrix} 0.0425 & 0.0242 & 0.0402 & 0.1600 \\ 0.0242 & 0.0239 & 0.0302 & 0.1200 \\ 0.0402 & 0.0302 & 0.0665 & 0.1999 \\ 0.1600 & 0.1200 & 0.1999 & 0.7999 \end{Vmatrix},$$

$$H_5 = \begin{Vmatrix} 0.0427 & 0.0240 & 0.0401 & 0.1600 \\ 0.0240 & 0.0240 & 0.0300 & 0.1200 \\ 0.0401 & 0.0300 & 0.0666 & 0.2000 \\ 0.1600 & 0.1200 & 0.2000 & 0.8000 \end{Vmatrix}$$

The components of the matrix H_1 differ from the components of the matrix H_5 by less than 1%. The classical Poisson inertia tensor $I(A)$ is calculated using the Poinsot tensor K_0 which has been found using the formula $I(A) = \mathrm{tr}(K_0)E - K_0$, and the Huygens formula

$$I(G) = I(A) - m[(\overline{AG}^T\overline{AG})E - \overline{AG}\,\overline{AG}^T]$$

was used to transfer from the inertia tensor at the point A to the central inertia tensor at the centre of mass G.

The inertia parameters of the parallelepiped which have been identified are identical to the specified parameters up to four decimal places:

$$m = 0.8 \text{ kg}, \quad \overline{AG} = [0.20\ 0.15\ 0.25]^T \text{ m},$$
$$I(G) = \mathrm{diag}(0.02267,\ 0.02733,\ 0.1667) \text{ m}^2\text{kg}$$

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